

The Hypermultiplet with Heisenberg Isometry in $N = 2$ Global and Local Supersymmetry

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Abstract

The string coupling of $N = 2$ supersymmetric compactifications of type II string theory on a Calabi-Yau manifold belongs to the so-called universal dilaton hypermultiplet, that has four real scalars living on a quaternion-Kähler manifold. Requiring Heisenberg symmetry, which is a maximal subgroup of perturbative isometries, reduces the possible manifolds to a one-parameter family that describes the tree-level effective action deformed by the only possible perturbative correction arising at one-loop level. A similar argument can be made at the level of global supersymmetry where the scalar manifold is hyper-Kähler. In this work, the connection between global and local supersymmetry is explicitly constructed, providing a non-trivial gravity decoupled limit of type II strings already in perturbation theory.

1 Introduction

Type II superstring theories compactified on a Calabi-Yau threefold (CY₃) yield $N = 2$ supersymmetry in four dimensions. Even if gauge symmetries are absent in perturbation theory, the study of the effective field theory plays an important role in moduli stabilization by fluxes, as well as in a more realistic framework, such as in the presence of orientifolds and D-branes. Besides supergravity fields, their massless spectra involve vector multiplets and hypermultiplets describing all Kähler class and complex structure deformations of the CY₃ manifold, as well as a universal hypermultiplet containing the string dilaton. Because of its special connection to the string coupling, the study of this universal hypermultiplet is an important problem per se. In the following, we restrict ourselves to this sector which also becomes exact in the particular case of a compactification with no Kähler class (complex structure) moduli in type IIB (IIA), or when these closed string moduli are fixed.

The dilaton hypermultiplet contains four real scalars parametrizing a quaternion-Kähler manifold, as required by $N = 2$ supergravity [1]. Two of them come from the NS-NS (Neveu-Schwarz) sector and correspond to the string dilaton (associated to the string coupling constant) and the universal axion, Poincaré dual of the antisymmetric tensor $B_{\mu\nu}$. The other two come from the R-R (Ramond) sector and are obtained from various n -form gauge potentials. On the IIB side, they correspond to another scalar (0-form) C_0 and the dual of the two-form $C_{\mu\nu}$. At the string tree-level, these four scalars live on the symmetric coset $SU(1,2)/SU(2) \times U(1)$ which is also Kähler [2]. Perturbative string corrections keep at least three isometries corresponding to the three independent shifts of the NS-NS axion and the R-R scalars, generating the Heisenberg algebra. Imposing just these isometries and the quaternion-Kähler structure, one finds that the only possible perturbative correction arises at one loop, destroying the Kähler structure of the manifold [3]. This correction was computed in [4, 3] and was found to be proportional to the Euler number of the CY₃.

More precisely, in the context of IIB superstrings, the Heisenberg algebra is generated by a combination of the gauge symmetries of the two antisymmetric tensors $B_{\mu\nu}$ (NS-NS) and $C_{\mu\nu}$ (R-R) and of the shift symmetry of the R-R scalar C_0 :

$$\delta B_{\mu\nu} = 2 \partial_{[\mu} \Lambda_{\nu]}, \quad \delta C_{\mu\nu} = 2 \partial_{[\mu} \tilde{\Lambda}_{\nu]} + \lambda B_{\mu\nu}, \quad \delta C_0 = \lambda. \quad (1.1)$$

As a consequence, the theory depends on the invariant three-forms

$$H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]}, \quad F_{\mu\nu\rho} = 3 \partial_{[\mu} C_{\nu\rho]} - C_0 H_{\mu\nu\rho} \quad (1.2)$$

and on $\partial_\mu C_0$. The Heisenberg algebra follows from

$$[\delta_1, \delta_2] C_{\mu\nu} = 2 \partial_{[\mu} \lambda_2 \Lambda_{1\nu]} - 2 \partial_{[\mu} \lambda_1 \Lambda_{2\nu]}. \quad (1.3)$$

After reduction to four dimensions, the gauge symmetries imply that each tensor can be dualized into a scalar field with axionic shift symmetry. The third global symmetry (with parameter λ) combines then with the axionic shifts to realize again the Heisenberg algebra on three scalar fields.

Indeed, one obtains three scalar fields φ , τ and $\eta = C_0$, with Heisenberg variations

$$\delta\eta = c_X, \quad \delta\varphi = c_Y, \quad \delta\tau = c_Z - c_X\varphi. \quad (1.4)$$

The scalars φ and τ are Poincaré dual to $C_{\mu\nu}$ and $B_{\mu\nu}$, respectively. The duality relations are, schematically,

$$\partial_\mu\varphi \sim \epsilon_{\mu\nu\lambda\rho}F^{\nu\lambda\rho}, \quad \partial_\mu\tau + \eta\partial_\mu\varphi \sim \epsilon_{\mu\nu\lambda\rho}H^{\nu\lambda\rho}.$$

The algebra is $[X, Y] \sim Z$, with Y and Z generating the axionic shifts (with parameters c_Y and c_Z), while X generates the shift of the R-R scalar (with parameter c_X). Notice that the central charge of the algebra is (depending on the representation) the gauge symmetry of the R-R tensor and the axionic symmetry of τ , dual to the NS-NS tensor.

Actually, as we will see later on, the Heisenberg algebra is extended by a fourth perturbative generator M that rotates X, Y and commutes also with the central charge Z :

$$\delta_M\eta = c_M\varphi, \quad \delta_M\varphi = -c_M\eta, \quad \delta_M\tau = \frac{c_M}{2}(\eta^2 - \varphi^2). \quad (1.5)$$

Equivalently, M rotates the phase of the complex R-R scalar $\eta + i\varphi$. As a result, the perturbative symmetry becomes the two-dimensional Euclidean group E_2 with central extension Z .

Imposing $N = 2$ supersymmetry *and* Heisenberg symmetry is a powerful constraint. In a previous work [5], we have briefly analyzed its implications in global $N = 2$ supersymmetry, where hypermultiplet scalars form a Ricci-flat hyper-Kähler manifold [6]. For a single hypermultiplet, we found a unique non-trivial hyper-Kähler space admitting a one-parameter deformation. In local supersymmetry where hypermultiplets live on quaternion-Kähler manifolds [1], a one-parameter family of solutions emerges [3] from the general analysis of Calderbank and Pedersen [7]. These similar results suggest a correspondence between the local and global cases which could be studied using a Ricci-flat limit of the quaternion-Kähler manifold preserving the Heisenberg algebra. This is the main goal of the present article.

Taking the limit $\kappa \rightarrow 0$ in a hypermultiplet theory coupled to $N = 2$ supergravity is a subtle problem. In contrast to the simplest case of $N = 1$, $N = 2$ supergravity imposes that the curvature of the hypermultiplet scalar manifold is proportional to the gravitational coupling κ^2 and hence the curved, quaternion-Kähler Einstein metric

of the local hypermultiplet smoothly turns to a Ricci-flat hyper-Kähler metric. However, to obtain a non-trivial space, an appropriate limit must be defined, involving a new mass scale that should remain finite as Planck mass goes to infinity. This mechanism has only been explicitly displayed for some particular cases, mostly using the quaternionic quotient method [8, 9].

Thus, in this work, we establish the precise connection between the local and global $N = 2$ supersymmetric actions of a single hypermultiplet with Heisenberg isometry, to explicitly obtain the gravity-decoupled limit. We first reconstruct the $N = 2$ supergravity Lagrangian using the method of quaternionic quotient, starting with superconformal supergravity and imposing gauge conditions and constraints. We then define the zero-curvature limit that reduces the perturbative-corrected metric of the dilaton hypermultiplet to the non-trivial hyper-Kähler form found in [5]. It turns out that the deformation parameter corresponding to the one-loop correction plays a crucial role. Indeed, the zero-curvature limit of the tree level $SU(1, 2)/SU(2) \times U(1)$ metric is trivial, leading to free kinetic terms. The presence of the one-loop parameter however allows for a non-trivial limit, giving rise to a hyper-Kähler metric that depends on a mass scale which remains finite as the four-dimensional Planck mass goes to infinity. At the same time, the value of the string coupling is tuned to a fixed value determined by the one-loop correction and can be made weak for large and positive Euler number of the CY_3 manifold, so that non-perturbative corrections remain suppressed while taking the gravity decoupled limit.

This paper is organized as follows. In Section 2, we recall the global construction of a four-dimensional hyper-Kähler manifold with the Heisenberg isometry in three formulations [5]: single-tensor which has the advantage of an off-shell $N = 2$ supersymmetry formulation, scalar that provides a geometric description with a metric, and double tensor corresponding to the type IIB string basis¹. In Section 3, we first review the Calderbank-Pedersen metric with Heisenberg symmetry and show that the latter is actually extended by a fourth generator which commutes with its central charge and rotates the other two. This generates a fourth perturbative isometry of the metric, as described above. We then rederive in supergravity the quaternion-Kähler metric with the Heisenberg isometry [3], by taking an appropriate quaternionic quotient of the symmetric quaternion-Kähler space $Sp(2, 4)/Sp(2) \times Sp(4)$ containing two hypermultiplets. Their reduction to one is achieved by gauging a symmetry corresponding to the central charge of the Heisenberg algebra. In Section 4, we take the zero-curvature limit, leading to the one parameter hyper-Kähler manifold with Heisenberg symmetry of Section 2. We thus find the correspondence of string fields in the rigid globally

¹Note though that the basis of string vertex operators corresponds to the single-tensor representation [3].

supersymmetric limit and we also discuss the coupling of the dilaton hypermultiplet to a D-brane where one of the two supersymmetries is non-linearly realized. Finally, Section 5 contains some concluding remarks.

2 On the Heisenberg algebra and global supersymmetry

2.1 Lagrangians

Consider a $N = 1$ globally supersymmetric theory with two superfields, a chiral Φ and a real linear L . It contains three real scalars, $\text{Re } \phi = \text{Re } \Phi|_{\theta=0}$, $\text{Im } \phi = \text{Im } \Phi|_{\theta=0}$, and $C = L|_{\theta=0}$, and L also depends on the curl of an antisymmetric tensor $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$. The Lagrangian (up to two derivatives) is

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \mathcal{H}(L, \Phi, \bar{\Phi}) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}). \quad (2.1)$$

Besides the gauge invariance of $B_{\mu\nu}$ which does not act on the superfields, we also impose a two-parameter global symmetry acting on Φ with variations

$$\delta\Phi = \alpha - i\beta. \quad (2.2)$$

In this formulation, all three symmetries trivially commute. Nevertheless, in the version where $B_{\mu\nu}$ is dualized to a scalar, or in the version where $\text{Im } \phi$ (for instance) is transformed into a second antisymmetric tensor, the three-parameter symmetry realizes a Heisenberg algebra acting either on three scalars according to Eq. (1.4), as in the hypermultiplet formulation of IIB strings compactified to four dimensions, or on two tensors and one scalar according to Eqs. (1.1) and (1.3). The Lagrangian compatible with the required symmetry (2.2) has

$$\mathcal{H}(L, \Phi, \bar{\Phi}) = \mathcal{F}(L) + [AL + B]\Phi\bar{\Phi}, \quad W(\Phi) = k\Phi, \quad (2.3)$$

with an arbitrary function $\mathcal{F}(L)$ and real constants A and B .² The constant k generates a C -dependent potential $V = |k|^2/(AC + B)$ which does not admit a vacuum if $A \neq 0$. We take then $k = 0$.

The superfields Φ and L provide an off-shell representation of the $N = 2$ single-tensor multiplet. On the $N = 1$ Lagrangian, the condition for a second supersymmetry is [10]³

$$\frac{\partial^2 \mathcal{H}}{\partial L^2} + 2 \frac{\partial^2 \mathcal{H}}{\partial \Phi \partial \bar{\Phi}} = 0, \quad (2.4)$$

²Of course, B can be eliminated by a constant shift of L .

³The same conventions as in Ref. [5] are used. They slightly differ from Ref. [10].

which in turn indicates that

$$\mathcal{F}_{N=2}(L) = -\frac{A}{3}L^3 - BL^2. \quad (2.5)$$

The same theory is given by

$$\widehat{\mathcal{F}}_{N=2}(L) = -\frac{1}{3A^2}(AL + B)^3. \quad (2.6)$$

Hence, the $N = 2$ theory compatible with complex shift symmetry of Φ is the sum

$$\mathcal{L}_{N=2} = \int d^2\theta d^2\bar{\theta} \left[A \left(-\frac{1}{3}L^3 + L\Phi\bar{\Phi} \right) + B(-L^2 + \Phi\bar{\Phi}) \right] \quad (2.7)$$

of a trilinear interacting term and of a free term where the symmetry is trivial. If canonical dimensions are assigned to L and Φ , A has dimension $(\text{mass})^{-1}$ and B is dimensionless.

For further use, we need the bosonic component expansion of this superfield theory. Using

$$\begin{aligned} L(x, \theta, \bar{\theta}) &= C + \theta\sigma^\mu\bar{\theta}v_\mu + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square C, & v_\mu &= \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma}, & H_{\nu\rho\sigma} &= 3\partial_{[\nu}B_{\rho\sigma]}, \\ \Phi(x, \theta, \bar{\theta}) &= \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \theta\theta f - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square\phi, \end{aligned}$$

we obtain⁴

$$\begin{aligned} \mathcal{L}_{N=2, \text{bos.}} &= (AC + B) \left[\frac{1}{2}(\partial_\mu C)^2 + (\partial_\mu\phi)(\partial^\mu\bar{\phi}) + \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} \right] \\ &\quad - \frac{i}{12}A\epsilon^{\mu\nu\rho\sigma}(\bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi})H_{\nu\rho\sigma}. \end{aligned} \quad (2.8)$$

Since, $\partial_{[\mu}H_{\nu\rho\sigma]} = 0$, the variation (2.2) of ϕ induces a total derivative. Kinetic terms are positive if $AC + B > 0$. If $A \neq 0$, B can be eliminated by shifting C . The (shifted) field C will be assumed strictly positive and the two options are an interacting, cubic theory with $A > 0$ and $B = 0$, or the free theory $A = 0$, $B > 0$.

We may then perform two supersymmetric duality transformations [11] on theory (2.3), either turning the linear L into a chiral S or turning the chiral Φ into a second linear multiplet L' . The first transformation leads to

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left[\tilde{\mathcal{F}}(\mathcal{Y}) + B\Phi\bar{\Phi} \right], \quad (2.9)$$

where $\tilde{\mathcal{F}}(\mathcal{Y})$ is the Legendre transform of $\mathcal{F}(L)$ and the variable is⁵ $\mathcal{Y} = S + \bar{S} + A\Phi\bar{\Phi}$. Invariance of \mathcal{Y} under shift symmetries (2.2) requires a compensating variation of S :

$$\delta_H S = (\alpha\delta_X + \beta\delta_Y + \gamma\delta_Z)S = -A(\alpha + i\beta)\Phi + 2i\gamma, \quad (2.10)$$

⁴The auxiliary field f vanishes.

⁵Notice that $\int d^2\theta d^2\bar{\theta}\Phi\bar{\Phi} = \frac{1}{A}\int d^2\theta d^2\bar{\theta}\mathcal{Y} + \text{derivative}$.

where the axionic shift symmetry of $\text{Im } S$ is dual to the gauge symmetry of $B_{\mu\nu}$, and the subscripts X, Y, Z make clear the correspondence with the transformations (1.4). Indeed, since

$$[\delta'_H, \delta_H]S \equiv -A(\alpha' + i\beta')\delta_H\Phi + A(\alpha + i\beta)\delta'_H\Phi = 2iA(\alpha'\beta - \alpha\beta'), \quad [\delta_H, \delta'_H]\Phi = 0, \quad (2.11)$$

the chiral theory has Heisenberg symmetry. Moreover, the theory (2.9) has another symmetry M rotating the chiral superfield Φ , as already mentioned in the Introduction (see Eq. (1.5)).

For the $N = 2$ single-tensor theory (2.7), the dual hypermultiplet theory⁶ is

$$\mathcal{L}_{N=2} = \int d^2\theta d^2\bar{\theta} \mathcal{K}(\mathcal{Y}) = \frac{2}{3A^2} \int d^2\theta d^2\bar{\theta} (A\mathcal{Y} + B^2)^{3/2}. \quad (2.12)$$

Eliminating some derivatives, the limiting case $A = 0$ is a free theory. As required for a hyper-Kähler sigma-model, the determinant of the Kähler metric is constant (and positive).

A useful change of variable is

$$\hat{S} = S - \frac{A}{2}\Phi^2, \quad \mathcal{Y} = \hat{S} + \bar{\hat{S}} + \frac{A}{2}(\Phi + \bar{\Phi})^2. \quad (2.13)$$

and transformation (2.10) becomes $\delta_H\hat{S} = -2A\alpha\Phi + 2i\gamma$. With these variables, the transformations with parameters β and γ only act as shift symmetries of $\text{Im } \Phi$ and $\text{Im } \hat{S}$ respectively. In terms of variables \mathcal{Y} , $\text{Im } \hat{S}$, $\text{Re } \Phi$ and $\text{Im } \Phi$, one immediately deduces that the most general Heisenberg-invariant supersymmetric theory is of the form (2.9).

Performing the second duality transformation of the chiral Φ into a linear L' , always leads to the dual theory

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left[\mathcal{F}(L) - \frac{1}{2} \frac{L'^2}{AL + B} \right], \quad (2.14)$$

with \mathcal{F} given in Eq. (2.5). Expression (2.14) is actually the most general $N = 1$ Lagrangian for L and L' with symmetry

$$\delta L' = \alpha(AL + B). \quad (2.15)$$

This transformation, which links the two antisymmetric tensors in L and L' as in variation (1.1), forms with their respective gauge symmetries a Heisenberg algebra realized as in type IIB strings.

⁶With positive Kähler metric.

Instead of $\text{Im } \Phi$, we could have chosen to dualize $e^{ia}\Phi$ for any phase a , since

$$\int d^2\theta d^2\bar{\theta} (AL + B)\Phi\bar{\Phi} = \frac{1}{2} \int d^2\theta d^2\bar{\theta} (AL + B)(e^{ia}\Phi + e^{-ia}\bar{\Phi})^2 + \text{derivative}.$$

The result would be again theory (2.14). This is a consequence of symmetry M , which is however fixed by the choice of dualization and does not act on L' .

2.2 Hyper-Kähler metrics with Heisenberg symmetry

The Kähler coordinates defined by $N = 1$ chiral superfields S and Φ are not necessarily the most appropriate to describe a hyper-Kähler manifold. There is a ‘standard’ set of coordinates used to describe hyper-Kähler metrics with shift isometries in the literature. For comparison purposes, we define in this Subsection these coordinates in terms of our superfield components.

For any hyper-Kähler manifold with a shift symmetry, one can find coordinates in which the metric has the Gibbons-Hawking form [12]

$$ds^2 = f(\vec{x}) dx_i dx_i + f(\vec{x})^{-1} (d\tau + \omega_i dx_i)^2, \quad (2.16)$$

with condition $\vec{\nabla} \times \vec{\omega} = \vec{\nabla} f$. Imposing the requirement of a Heisenberg symmetry acting according to

$$\delta_H x_1 = \sqrt{2}\alpha, \quad \delta_H x_2 = -\sqrt{2}\beta, \quad \delta_H x_3 = 0, \quad \delta_H \tau = -\sqrt{2}\alpha x_2 + \gamma \quad (2.17)$$

also defines $d\tau + x_1 dx_2$ as the invariant derivative of τ and indicates that $\vec{\omega} = (0, x_1, 0)$. The value of $f(\vec{x})$ follows then from $\vec{\nabla} \times \vec{\omega} = \vec{\nabla} f$. This last condition is invariant under $\vec{\omega} \rightarrow \vec{\omega} + \vec{\nabla} \lambda(\vec{x})$, for any gauge function $\lambda(\vec{x})$. In turn, invariance of the metric requires the compensating transformation $\tau \rightarrow \tau - \lambda(\vec{x})$.

From the $N = 2$ Kähler potential (2.12), the Kähler metric can be written⁷

$$ds^2 = \frac{1}{2}(A\mathcal{Y} + B^2)^{-1/2} \left[\frac{1}{4}d\mathcal{Y}^2 + \left(d\text{Im } S + i\frac{A}{2}(\Phi d\bar{\Phi} - \bar{\Phi} d\Phi) \right)^2 \right] + (A\mathcal{Y} + B^2)^{1/2} d\Phi d\bar{\Phi}, \quad (2.18)$$

using coordinates $(\mathcal{Y}, \text{Im } S, \text{Re } \Phi, \text{Im } \Phi)$. The supersymmetric duality transformation from L to S exchanges a real scalar $C = L|_{\theta=0}$, invariant under Heisenberg variations, and $\text{Re } S$ with variation (2.10). The Legendre transformation defines the change of variable from \mathcal{Y} to C :

$$AC + B = \sqrt{A\mathcal{Y} + B^2}. \quad (2.19)$$

⁷From here on, we do not distinguish chiral superfields S and Φ and their lowest complex scalar components.

Then, in terms of coordinates $(C, \text{Im } S, \text{Re } \Phi, \text{Im } \Phi)$, the metric becomes

$$ds^2 = \frac{AC + B}{2} [dC^2 + 2 d\Phi d\bar{\Phi}] + \frac{2}{(AC + B)} \left(d\tau + A \text{Re } \Phi d\text{Im } \Phi \right)^2. \quad (2.20)$$

This is the Gibbons-Hawking metric (2.16) with $\vec{x} = (\sqrt{2} \text{Re } \Phi, \sqrt{2} \text{Im } \Phi, C)$ and

$$\tau = \frac{1}{2} (\text{Im } S - A \text{Re } \Phi \text{Im } \Phi) = \frac{1}{2} \text{Im } \hat{S}.$$

The function

$$f(\vec{x}) = \frac{AC + B}{2} \quad (2.21)$$

solves the hyper-Kähler condition $\vec{\nabla} \times \vec{\omega} = \vec{\nabla} f$ with $\vec{\omega} = (0, \frac{A}{2} x_1, 0)$. Choosing for instance $\lambda = -\frac{A}{2} x_1 x_2$ turns then $\vec{\omega}$ into $(-\frac{A}{2} x_2, 0, 0)$ and $d\tau + \frac{A}{2} x_1 dx_2$ into $d\tau - \frac{A}{2} x_2 dx_1$. Similarly, a rotation of Φ

$$\delta_M x_1 = m x_2, \quad \delta_M x_2 = -m x_1,$$

which is compatible with the shift symmetry (2.2), corresponds to $\lambda(\vec{x}) = \frac{Am}{4} (x_2^2 - x_1^2)$. It is the isometry M of metric (2.20).

The conclusion is that the Gibbons-Hawking Ansatz for the hyper-Kähler metric corresponds to coordinates where $\text{Re } S$ is replaced by its Legendre dual C , which is also the lowest scalar component of the linear superfield dual to S .

3 The universal hypermultiplet in $N = 2$ supergravity

Hypermultiplet scalars of $N = 2$ supergravity live on $4n$ -dimensional quaternion-Kähler manifolds with holonomy included in $Sp(2n) \times Sp(2)$. Supergravity requires that the curvature of these Einstein spaces is proportional to the gravitational coupling κ^2 [1]. Hence, the decoupling limit $\kappa \rightarrow 0$ turns the hypermultiplet manifold into a Ricci-flat hyper-Kähler space, as required by global $N = 2$ supersymmetry [6]. For a single hypermultiplet, or a four-dimensional quaternion-Kähler manifold, the defining condition on the holonomy is not pertinent since $Sp(2) \times Sp(2) \sim SO(4)$. The relevant condition is then self-duality of the Weyl tensor.

3.1 The Calderbank-Pedersen metric with Heisenberg symmetry

Calderbank and Pedersen [7] have classified all four-dimensional Einstein metrics with self-dual Weyl curvature and two commuting isometries. Using coordinates $(\rho, \eta, \varphi, \tau)$

with the isometries acting as shifts of φ and τ , their metrics are written in terms of any single function $F(\rho, \eta)$ verifying

$$\frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \eta^2} = \frac{3F}{4\rho^2}. \quad (3.1)$$

It is simple to see [3] that metrics with Heisenberg symmetry are then obtained if F does not depend on η , *i.e.* if⁸

$$\sqrt{\rho} F(\rho) = \frac{1}{2}[\rho^2 - \chi], \quad (3.2)$$

with an arbitrary real parameter χ . The Calderbank-Pedersen metric with Heisenberg symmetry (the CPH metric) reads then

$$ds_{CPH}^2 = \frac{\rho^2 + \chi}{(\rho^2 - \chi)^2} (d\rho^2 + d\eta^2 + d\varphi^2) + \frac{4\rho^2}{(\rho^2 - \chi)^2(\rho^2 + \chi)} (d\tau + \eta d\varphi)^2. \quad (3.3)$$

The coordinate ρ is positive, $\rho > 0$, and positivity of the metric requires $\rho^2 + \chi > 0$, a stronger condition if χ is negative. It is an Einstein metric with negative curvature, and is Kähler only if $\chi = 0$. Notice that if $\chi \neq 0$, the rescaling $(\rho, \eta, \varphi, \tau) \rightarrow (|\chi|^{1/2}\rho, |\chi|^{1/2}\eta, |\chi|^{1/2}\varphi, |\chi|\tau)$ turns χ in metric (3.3) into ± 1 . This is not true if we turn on string interactions, such as in the presence of D-branes where the dilaton, or equivalently the field ρ , couples to the Dirac-Born-Infeld (DBI) action in a non-trivial way (see Section 4). For this reason, we keep explicitly χ throughout the paper. We may use a new coordinate $V = \rho^2$ with metric

$$ds_{CPH}^2 = \frac{V + \chi}{(V - \chi)^2} \left(\frac{dV^2}{4V} + d\eta^2 + d\varphi^2 \right) + \frac{4V}{(V - \chi)^2(V + \chi)} (d\tau + \eta d\varphi)^2. \quad (3.4)$$

The particular case $\chi = 0$ has extended symmetry: it is the $SU(2, 1)/SU(2) \times U(1)$ metric with Kähler potential

$$K(\hat{S}, \bar{\hat{S}}, \Phi, \bar{\Phi}) = -\ln V, \quad V = \hat{S} + \bar{\hat{S}} - (\Phi + \bar{\Phi})^2, \quad (3.5)$$

and with $\Phi = \frac{1}{\sqrt{2}}(\eta + i\varphi)$, $\tau = -\frac{1}{2} \text{Im } \hat{S}$.

The CPH metric is invariant under four isometry variations acting on coordinates (η, φ, τ) :

$$\begin{aligned} \delta_X \eta &= \sqrt{2}, & \delta_Y \eta &= 0, & \delta_Z \eta &= 0, & \delta_M \eta &= \varphi, \\ \delta_X \varphi &= 0, & \delta_Y \varphi &= -\sqrt{2}, & \delta_Z \varphi &= 0, & \delta_M \varphi &= -\eta, \\ \delta_X \tau &= -\sqrt{2}\varphi, & \delta_Y \tau &= 0, & \delta_Z \tau &= 1, & \delta_M \tau &= \frac{1}{2}(\eta^2 - \varphi^2). \end{aligned} \quad (3.6)$$

⁸The metric does not make sense without the $\rho^{3/2}$ contribution to F and the overall normalization of F is a choice of coordinates. Our χ is $\hat{\chi}$ in Ref. [3].

The non-zero commutators are

$$[X, Y] = 2Z, \quad [M, X] = Y, \quad [M, Y] = -X. \quad (3.7)$$

Hence, X , Y and Z generate the Heisenberg algebra and Z is a central extension of a two-dimensional euclidean algebra generated by M (which rotates φ and η), X and Y (which translate φ and η). With these conventions,

$$\delta_H \Phi = (\alpha X + \beta Y + \gamma Z)\Phi = \alpha - i\beta, \quad \delta_H \hat{S} = 4\alpha \Phi - 2i\gamma \quad (3.8)$$

and V is invariant.

The metric (3.4) appears in the one-loop-corrected Lagrangian of the universal hypermultiplet of type II strings, reduced to four dimensions, with the NS-NS and R-R tensors dualized to scalars with shift symmetry [3]. At one-loop order, the four-dimensional dilaton field is related to coordinate V and parameter χ by

$$e^{-2\phi_4} = V - \chi, \quad \chi = -\chi_1, \quad \chi_1 = \frac{\chi_E}{12\pi}, \quad (3.9)$$

where χ_E is the Euler number of the internal CY_3 manifold. The real number χ_1 encodes the one-loop correction [3]. Notice that this relation also indicates that $V - \chi = V + \chi_1 > 0$, which is stronger than $V = \rho^2 > 0$ if the Euler number is negative ($\chi > 0$). Since positivity of the CPH metric also requires $V + \chi > 0$ if $\chi < 0$, the domain of V is naturally restricted to $V > |\chi|$.

The R-R scalar is

$$C_0 \equiv \eta, \quad (3.10)$$

and is shifted by symmetry X . Finally, Poincaré duality gives the following equivalences

$$\begin{aligned} d\varphi &\sim F_3 = dC_2 - \eta dB_2, \\ d\tau + \eta d\varphi &\sim H_3 = dB_2. \end{aligned}$$

In the scalar version, the central charge is the shift Z of τ (related to the NS-NS tensor B_2) while in the two-tensor version, it is the gauge variation of the (R-R) tensor C_2 . Writing η and φ in a complex Φ is conventional: we always use

$$\Phi = \frac{1}{\sqrt{2}}(\eta + i\varphi).$$

In the previous Section, we found a unique four-dimensional hyper-Kähler manifold with Heisenberg symmetry. It also admits the fourth isometry M rotating Φ . In the quaternion-Kähler case, the theorem of Calderbank-Pedersen [7] leads then to a very similar uniqueness conclusion. We will see how these two results are connected when taking an appropriate zero-curvature limit. But we first want to obtain the $N = 2$ supergravity coupling of the universal hypermultiplet on the CPH manifold.

3.2 Coupling to $N = 2$ supergravity

There are different methods to construct hypermultiplet couplings to $N = 2$ supergravity. The simplest procedure, which is however not the most general, is to use hypermultiplets coupled to local $N = 2$ superconformal symmetry [13] and to perform a quaternionic quotient [8, 9] using supplementary hypermultiplet(s) and non-propagating vector multiplet(s). In this Section, we use this procedure to obtain the supergravity theory of the one-loop-corrected dilaton hypermultiplet.

Related constructions, using more general but also more complicated methods, can be found in Ref. [14], in the language of projective superspace or in Ref. [15], using harmonic superspace.

Conformal $N = 2$ supergravity is the gauge theory of $SU(2, 2|2)$, which has a $SU(2)_R \times U(1)_R$ R -symmetry with non-propagating gauge fields. Pure Poincaré $N = 2$ supergravity is obtained from the superconformal coupling of one propagating vector multiplet⁹ (which may be charged under $U(1)_R$) and one hypermultiplet (charged under $SU(2)_R$) by gauge-fixing of the extraneous symmetries. These two multiplets include in particular the compensating fields used in the gauge-fixing to the Poincaré theory.

For the superconformal construction of our particular hypermultiplet sigma-model, we also need a physical hypermultiplet, with positive kinetic metric, to describe the dilaton multiplet. In addition, for the quaternionic quotient, we need a non-propagating vector multiplet with gauge field W_μ , gauging a specific generator T to be discussed below, and, since the elimination of the algebraic vector multiplet involves three constraints and one gauge choice on scalar fields, we also need a third non-physical hypermultiplet. Its kinetic metric can have a positive or negative sign, depending on the constraints induced by the choice of T . Hence, we need to consider the $N = 2$ superconformal theory of two vector multiplets and three hypermultiplets. The superconformal hypermultiplet scalar sector has then an ‘automatic’ $Sp(2, 4)$ global symmetry in which the gauge generator T of the quaternionic quotient is chosen.

3.3 $Sp(2, 4)$

In the following, we consider three hypermultiplets coupled to (superconformal) $N = 2$ supergravity. One (compensating) hypermultiplet has negative signature, the physical hypermultiplet has positive signature, the third hypermultiplet, associated to the non-propagating vector multiplet, may have a positive or negative signature, depending on the constraints applied to the scalar fields. In any case, we are considering $Sp(2, 4)$ –

⁹Its gauge field is the graviphoton.

invariant supergravity couplings of $N = 2$ hypermultiplets.

The hypermultiplet scalars are A_i^α , with $SU(2)_R$ index $i = 1, 2$ and $Sp(2, 4)$ index $\alpha = 1, \dots, 6$. They transform in representation $(\mathbf{6}, \mathbf{2})$ of $Sp(2, 4) \times SU(2)_R$. Their conjugates are¹⁰

$$A_\alpha^i = (A_i^\alpha)^* = \epsilon^{ij} \rho_{\alpha\beta} A_j^\beta \quad (3.11)$$

with $\rho^{\alpha\beta} \rho_{\beta\gamma} = -\delta_\gamma^\alpha$ and $\epsilon^{ij} \epsilon_{jk} = -\delta_k^i$. We choose the $Sp(2, 4)$ -invariant metric as

$$\rho = I_3 \otimes i\sigma_2 = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \quad (3.12)$$

and we use

$$d = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}, \quad \eta = \text{diag}(-1, 1, -1), \quad \rho d \rho = -d. \quad (3.13)$$

In our choice of η , direction 1 corresponds to the superconformal compensator, direction 2 to the physical hypermultiplet and our choice of quaternionic quotient will require a negative metric in direction 3; otherwise, our construction does not work. On scalar fields, $Sp(2, 4)$ acts according to

$$\delta A_i^\alpha = g t^\alpha_\beta A_i^\beta, \quad \delta A_\alpha^i = g t_\alpha^\beta A_\beta^i, \quad t_\alpha^\beta = -\rho_{\alpha\gamma} t^\gamma_\delta \rho^{\delta\beta}. \quad (3.14)$$

Since relation (3.11) also implies $t_\alpha^\beta = (t^\alpha_\beta)^*$, the choice (3.12) and the invariance of $d_\beta^\alpha A_\alpha^i A_i^\beta$ lead to

$$t = \begin{pmatrix} U & \eta Q \\ -\eta Q^* & U^* \end{pmatrix}, \quad U^\dagger = -\eta U \eta, \quad Q = Q^\tau, \quad t^\dagger = -d t d. \quad (3.15)$$

This is an element of $Sp(2, 4)$: U generates the $U(1, 2)$ subgroup (9 generators) and Q (12 generators) generates $Sp(2, 4)/U(1, 2)$. The (2×2) matrix $A^\dagger d t A$, with matrix elements $A_\alpha^i d_\beta^\alpha t^\beta_\gamma A_j^\gamma$, is antihermitian, as required by gauge invariance of $A^\dagger d A$, and traceless.

3.4 The Heisenberg subalgebra of $SU(1, 2)$ and $Sp(2, 4)$

At string tree-level, the universal hypermultiplet of the dilaton in type II strings lives, when formulated in terms of four real scalars, on the quaternion-Kähler and Kähler manifold $SU(1, 2)/SU(2) \times U(1) = U(1, 2)/U(2) \times U(1)$. Since $U(1, 2) = SU(1, 2) \times U(1)_0$ is maximal in $Sp(2, 4)$, $Sp(2, 4)$ has a unique generator commuting with $SU(1, 2)$: the generator of $U(1)_0$. At one-loop however, the isometry is reduced and includes the

¹⁰We follow the conventions of the second paper of Ref. [13].

Heisenberg algebra which is known to be a subalgebra of $SU(1, 2)$. We need to find the most general generator T of $Sp(2, 4)$ which commutes with a Heisenberg subalgebra. In the following subsections, we will perform the quaternionic quotient construction induced by the gauging of T .

Since elements U of the $U(1, 2)$ algebra verify $U^\dagger = -\eta U \eta$ and we have chosen $\eta = \text{diag}(-1, 1, -1)$, a generic U is

$$U = \begin{pmatrix} ia & A & B \\ \bar{A} & ib & C \\ -\bar{B} & \bar{C} & ic \end{pmatrix}, \quad (3.16)$$

with a, b, c real, A, B, C complex and elements of $SU(1, 2)$ are traceless. On a three-dimensional complex vector, $U(1, 2)$ variations are $\delta A = UA$.

We may define the Heisenberg subalgebra as the $U(1, 2)$ transformations leaving $A_1 - A_2$ invariant: $(\delta_H A)_1 - (\delta_H A)_2 = (UA)_1 - (UA)_2 = 0$. The transformations acting on A_1 and A_2 are generated by the following three elements

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & i \\ i & -i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & -i & 0 \\ i & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.17)$$

which verify

$$0 = XZ = ZX = YZ = ZY = Z^2, \quad XY = -YX = Z, \quad X^2 = Y^2 = iZ. \quad (3.18)$$

The Heisenberg algebra

$$[X, Y] = 2Z, \quad [X, Z] = [Y, Z] = 0 \quad (3.19)$$

is then realized as a subalgebra of $SU(1, 2)$, with variations

$$\delta_H A = (\alpha X + \beta Y + \gamma Z) A = \begin{pmatrix} i\gamma & -i\gamma & \alpha + i\beta \\ i\gamma & -i\gamma & \alpha + i\beta \\ -\alpha + i\beta & \alpha - i\beta & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad (3.20)$$

in the fundamental representation. Since Z is a central charge of the Heisenberg algebra, we are interested in the elements of $U(1, 2)$ which commute with Z . They form an algebra generated by five elements, U_0, M, X, Y and Z , with

$$U_0 = iI_3, \quad M = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (3.21)$$

(U_0 generates the abelian factor of $U(1, 2) = SU(1, 2) \times U(1)_0$). Besides the Heisenberg algebra generated by X, Y, Z , we also have

$$[M, X] = 3Y, \quad [M, Y] = -3X \quad (3.22)$$

and M generates a rotation of (X, Y) leaving $X^2 + Y^2 = 2iZ$ invariant: $[M, X^2 + Y^2] = 2i[M, Z] = 0$.

One then easily checks that the most general $U(1, 2)$ generator which commutes with the Heisenberg algebra generated by X, Y, Z is proportional to

$$\hat{T} = U_0 + \chi Z = i \begin{pmatrix} 1 + \chi & -\chi & 0 \\ \chi & 1 - \chi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_0 = iI_3, \quad (3.23)$$

where χ is an arbitrary real number. If $\chi = 0$, $\hat{T} = U_0$ commutes with the whole $U(1, 2)$. If $\chi \neq 0$, \hat{T} commutes with the Heisenberg algebra supplemented by U_0 and M . The extension to $Sp(2, 4)$ is straightforward. Requiring that

$$T = \begin{pmatrix} \hat{T} & 0 \\ 0 & \hat{T}^* \end{pmatrix} \quad (3.24)$$

in $Sp(2, 4)$ commutes with an element of $Sp(2, 4)/U(1, 2)$ corresponds to find a (non-zero) symmetric matrix Q in Eq. (3.15) such that $\hat{T}^\dagger Q$ is also antisymmetric, which is impossible.¹¹ Hence, T is also the most general generator in $Sp(2, 4)$ which commutes with the Heisenberg algebra generated by X, Y and Z in $SU(1, 2)$. It actually commutes with X, Y, Z, M and U_0 .

3.5 $N = 2$ supergravity scalar Lagrangian

To construct the scalar kinetic metric, the relevant terms of the $N = 2$ conformal supergravity Lagrangian are [13, 8, 9]

$$\begin{aligned} e^{-1}\mathcal{L} &= d_\beta^\alpha (D_\mu A_i^\beta) (D^\mu A_\alpha^i) + (g d_\beta^\alpha A_\alpha^i T^\beta_\gamma A_k^\gamma Y_i^k + \text{c.c.}) \\ &\quad + \frac{1}{6} R (-X_0 \bar{X}_0 + d_\beta^\alpha A_\alpha^i A_i^\beta) + d (X_0 \bar{X}_0 + \frac{1}{2} d_\beta^\alpha A_\alpha^i A_i^\beta). \end{aligned} \quad (3.25)$$

The complex scalar X_0 is the partner of the graviphoton, $Y_j^i, Y_i^i = 0$, is the triplet of real auxiliary scalars in the non-propagating vector multiplet with gauge field W_μ used in the quaternionic quotient. The covariant derivatives are

$$\begin{aligned} D_\mu A_i^\alpha &= \partial_\mu A_i^\alpha - g' W_\mu T^\alpha_\beta A_i^\beta - g V_{\mu i}^j A_j^\alpha, \\ D_\mu A_\alpha^i &= \partial_\mu A_\alpha^i - g' W_\mu T_\alpha^\beta A_\beta^i - g V_{\mu j}^i A_\alpha^j, \end{aligned} \quad (3.26)$$

where g and g' are $SU(2)_R$ and $U(1)_T$ coupling constant. The (anti-hermitian) $SU(2)$ gauge fields $V_{\mu i}^j, V_{\mu i}^i = 0$, and the real auxiliary scalar d belong to the multiplet of superconformal gauge fields:

$$V_{\mu i}^j = \frac{i}{2} V_\mu^x (\sigma^x)_i^j, \quad V_{\mu j}^i = \epsilon^{ik} \epsilon_{jl} V_{\mu k}^l = (V_{\mu i}^j)^*.$$

¹¹This would not be true for $\hat{T} = Z$, which commutes with a larger subalgebra of $Sp(2, 4)$. The U_0 component is necessary.

We will commonly use a matrix notation, with a 6×2 complex matrix A and its 2×6 conjugate A^\dagger replacing A_i^α and A_α^i . Condition (3.11) implies that A contains six complex components only. It also implies, in particular, that $A^\dagger dA = \frac{1}{2} \text{Tr}(A^\dagger dA) I_2$. Since $V_\mu = -V_\mu^\dagger$, the Lagrangian and the derivatives read

$$\begin{aligned}
e^{-1} \mathcal{L} &= \text{Tr}(D_\mu A^\dagger) d(D^\mu A) + g \text{Tr} Y A^\dagger dTA + \text{c.c.} \\
&\quad + \frac{1}{6} R(-X_0 \overline{X}_0 + \text{Tr} A^\dagger dA) + d(X_0 \overline{X}_0 + \frac{1}{2} \text{Tr} A^\dagger dA); \\
D_\mu A &= \partial_\mu A - g' W_\mu T A - g V_\mu, \\
D_\mu A^\dagger &= \partial_\mu A^\dagger - g' W_\mu A^\dagger T^\dagger + g V_\mu A^\dagger.
\end{aligned} \tag{3.27}$$

Constraints are obtained from the elimination of the auxiliary fields and from the gauge-fixing of dilatation symmetry in the Poincaré theory:

- Einstein frame gauge-fixing condition and d auxiliary field equation:

$$X_0 \overline{X}_0 = \frac{1}{\kappa^2}, \quad \text{Tr} A^\dagger dA = -\frac{2}{\kappa^2}. \tag{3.28}$$

The second condition is invariant under $SU(2)_R$ and $Sp(4, 2)$. With an $SU(2)$ gauge choice, it allows to eliminate four scalar fields and would lead to the $Sp(4, 2)/Sp(4) \times Sp(2)$ sigma-model.

- Auxiliary fields Y_j^i :

$$A^\dagger dTA = 0. \tag{3.29}$$

Since this 2×2 matrix is traceless and antihermitian, these conditions eliminate three scalars and the associated abelian gauge invariance removes a fourth field.

The $SU(2)_R$ gauge fields $V_{\mu i}^j$ and the abelian W_μ have then algebraic field equations:

- Gauge field W_μ , associated with generator T :

$$W_\mu = \frac{\text{Tr}(\partial_\mu A^\dagger dTA - A^\dagger dT \partial_\mu A)}{2g' \text{Tr}(A^\dagger T^\dagger dTA)}. \tag{3.30}$$

- $SU(2)_R$ gauge fields $V_{\mu i}^j$:

$$V_\mu = -\frac{\partial_\mu A^\dagger dA - A^\dagger d\partial_\mu A}{g \text{Tr}(A^\dagger dA)}. \tag{3.31}$$

According to the second Eq. (3.28), the denominator is $-2g/\kappa^2$.

At this point, the scalar kinetic Lagrangian in theory (3.25) reduces to

$$\begin{aligned} e^{-1}\mathcal{L} &= e^{-1}(\mathcal{L}_{kin.} + \mathcal{L}_T + \mathcal{L}_{SU(2)}) \\ &= \text{Tr}(\partial_\mu A^\dagger) d(\partial^\mu A) - g'^2 \text{Tr}(A^\dagger T^\dagger dTA) W^\mu W_\mu - \frac{g^2}{\kappa^2} \text{Tr}(V^\mu V_\mu). \end{aligned} \quad (3.32)$$

The scalar fields are submitted to constraints (3.28) and (3.29) and the gauge fields W_μ and $V_{\mu i}^j$ are defined by their field equations (3.30) and (3.31).

To study the constraints (3.28) and (3.29) for our specific choice (3.23) and (3.24) of gauged generator T , we introduce two three-component complex vectors:

$$A_i^\alpha = \begin{pmatrix} \vec{A}_+ & \vec{A}_- \\ -\vec{A}_-^* & \vec{A}_+^* \end{pmatrix}, \quad A_\alpha^i = \begin{pmatrix} \vec{A}_+^* & \vec{A}_-^* \\ -\vec{A}_- & \vec{A}_+ \end{pmatrix}, \quad (3.33)$$

verifying the reality condition (3.11). On each doublet A_{+a} , A_{-a} , $a = 1, 2, 3$, act two different $SU(2)$ groups. Firstly, the superconformal $SU(2)_R$ acts on \pm indices. Secondly, $Sp(2, 4) \supset Sp(2)_1 \times Sp(2)_2 \times Sp(2)_3 \sim SU(2)_1 \times SU(2)_2 \times SU(2)_3$ and $(A_{+a}, -A_{-a}^*)$ is a doublet of $SU(2)_a$. One could define three quaternions

$$Q_a = \begin{pmatrix} A_{+a} & A_{-a} \\ -A_{-a}^* & A_{+a}^* \end{pmatrix} \quad a = 1, 2, 3 \quad (3.34)$$

with a left action of $SU(2)_a$ and a right action of the superconformal $SU(2)_R$. They verify (for each a)

$$Q_a Q_a^\dagger = Q_a^\dagger Q_a = \det Q_a I_2, \quad \det Q_a = |A_{+a}|^2 + |A_{-a}|^2. \quad (3.35)$$

The second condition (3.28) from $N = 2$ supergravity becomes:

$$\vec{A}_+^* \cdot \vec{A}_+ + \vec{A}_-^* \cdot \vec{A}_- = -\frac{1}{\kappa^2}, \quad \vec{A}^* \cdot \vec{A} = \vec{A}^\dagger \eta \vec{A} = -|A_1|^2 + |A_2|^2 - |A_3|^2. \quad (3.36)$$

With Eq. (3.24), condition (3.29) leads to three (real) equations:

$$\begin{aligned} \vec{A}_+^\dagger i\eta \hat{T} \vec{A}_+ &= \vec{A}_-^\dagger i\eta \hat{T} \vec{A}_-, \\ \vec{A}_-^\dagger i\eta \hat{T} \vec{A}_+ &= 0 \end{aligned} \quad (3.37)$$

($[i\eta \hat{T}]^\dagger = i\eta \hat{T}$). With the explicit form of \hat{T} , Eq. (3.23), and defining dimensionless fields $a_{\pm i} = \sqrt{2}\kappa A_{\pm i}$, the four constraints (3.36) and (3.37) read finally

$$\begin{aligned} I : \quad & |a_{+1}|^2 + |a_{-1}|^2 - |a_{+2}|^2 - |a_{-2}|^2 + |a_{+3}|^2 + |a_{-3}|^2 = 2, \\ II : \quad & -|a_{+1}|^2 + |a_{+2}|^2 - |a_{+3}|^2 - \chi |a_{+1} - a_{+2}|^2 \\ & = -|a_{-1}|^2 + |a_{-2}|^2 - |a_{-3}|^2 - \chi |a_{-1} - a_{-2}|^2, \\ III : \quad & 0 = -a_{+1}\bar{a}_{-1} + a_{+2}\bar{a}_{-2} - a_{+3}\bar{a}_{-3} - \chi(a_{+1} - a_{+2})(\bar{a}_{-1} - \bar{a}_{-2}). \end{aligned} \quad (3.38)$$

They are invariant under Heisenberg variations (3.20) of \vec{a}_+ and \vec{a}_- . The case $\chi = 0$ has been considered by Galicki [8]. Since it leads to $SU(1, 2)/SU(2) \times U(1)$, coordinates more appropriate for this larger isometry have been used.

3.6 Solving the constraints

To solve the constraints (3.38), we insist on keeping in \vec{a}_- a field Φ which transforms under the Heisenberg variations¹² $\delta_H \vec{a}_- = (\alpha X + \beta Y + \gamma Z) \vec{a}_-$ with a complex shift:

$$\delta_H \Phi = \alpha - i\beta. \quad (3.39)$$

This is the case if $a_{-1} = a_{-2}$, and a_{-3} is then invariant. We may define $\bar{\Phi} = a_{-1}/a_{-3}$ and constraint III reduces to $a_{+3} = (a_{+2} - a_{+1})\Phi$. Since

$$\delta_H \left(\frac{a_{+2} + a_{+1}}{a_{+2} - a_{+1}} \right) = -2i\gamma + 2(\alpha + i\beta) \frac{a_{+3}}{a_{+2} - a_{+1}} = -2i\gamma + 2\Phi \delta_H \bar{\Phi},$$

we finally define

$$S = \frac{a_{+2} + a_{+1}}{a_{+2} - a_{+1}} + Y, \quad \delta_H S = -2i\gamma + 2(\alpha + i\beta)\Phi \quad (3.40)$$

and the quantity

$$Y = S + \bar{S} - 2\Phi\bar{\Phi} \quad (3.41)$$

is invariant under Heisenberg variations. The algebra follows from $[\delta'_H, \delta_H] = (\alpha'\beta - \alpha\beta')[X, Y] = 2(\alpha'\beta - \alpha\beta')Z$:

$$[\delta'_H, \delta_H]S = 2(\alpha' + i\beta')\delta_H\Phi - 2(\alpha + i\beta)\delta'_H\Phi = -4i(\alpha'\beta - \alpha\beta') = 2(\alpha'\beta - \alpha\beta')Z.$$

These definitions are summarized in the choice

$$\vec{a}_- = \frac{K}{\Delta} \begin{pmatrix} \bar{\Phi} \\ \bar{\Phi} \\ 1 \end{pmatrix}, \quad \vec{a}_+ = \frac{1}{\Delta} \begin{pmatrix} S - Y - 1 \\ S - Y + 1 \\ a \end{pmatrix}, \quad (3.42)$$

with complex fields S , Φ and a . The four available gauge choices have been used to take $\Delta = |\Delta|$, $K = |K|$ and $a_{-1} = a_{-2}$. Under Heisenberg variations, Δ and K are invariant. Hence, we are left with eight real scalar fields submitted to the four constraints (3.38) which drastically simplify:

$$\begin{aligned} I : \quad & \Delta^2 \left(2 - |a_{+1}|^2 + |a_{+2}|^2 - |a_{+3}|^2 \right) = K^2, \\ II : \quad & 2(S + \bar{S}) - |a|^2 - 4Y = 4\chi - K^2, \\ III : \quad & a = 2\Phi. \end{aligned} \quad (3.43)$$

Hence, the solution is

$$\vec{a}_- = \sqrt{\frac{Y + 2\chi}{Y + \chi}} \begin{pmatrix} \bar{\Phi} \\ \bar{\Phi} \\ 1 \end{pmatrix}, \quad \vec{a}_+ = \frac{1}{\sqrt{2(Y + \chi)}} \begin{pmatrix} S - Y - 1 \\ S - Y + 1 \\ 2\Phi \end{pmatrix}. \quad (3.44)$$

¹²See Eq. (3.20).

The solution implies $Y + \chi > 0$ if $\chi > 0$ or $Y + 2\chi > 0$ if $\chi < 0$. The scalar kinetic Lagrangian (3.32) obtained from this solution is¹³

$$\begin{aligned}\kappa^2 \mathcal{L} = & \frac{(Y + 3\chi)}{4(Y + 2\chi)(Y + \chi)^2} (\partial_\mu Y)^2 - \frac{2}{Y + \chi} \partial_\mu \Phi \partial^\mu \bar{\Phi} \\ & + \frac{1}{2(Y + \chi)(Y + 3\chi)} [\text{Im}(\partial_\mu S - 2\bar{\Phi} \partial_\mu \Phi)]^2 \\ & + \frac{1}{2(Y + \chi)^2} [\text{Im}(\partial_\mu S - 2\bar{\Phi} \partial_\mu \Phi)]^2 + \frac{4(Y + 2\chi)}{(Y + \chi)^2} \partial_\mu \Phi \partial^\mu \bar{\Phi}.\end{aligned}\quad (3.45)$$

The first line comes from the basic scalar kinetic terms $\mathcal{L}_{kin.}$ in Lagrangian (3.32). The second line is the contribution \mathcal{L}_T of the gauge field of T , the third line arises from the supergravity $SU(2)_R$ gauge fields. Each term is separately invariant under Heisenberg variations. Collecting terms, the final form of the theory is

$$\begin{aligned}\kappa^2 \mathcal{L} = & \frac{Y + 3\chi}{(Y + \chi)^2} \left[\frac{1}{4} \frac{(\partial_\mu Y)^2}{Y + 2\chi} + 2 \partial_\mu \Phi \partial^\mu \bar{\Phi} \right] \\ & + \frac{Y + 2\chi}{(Y + 3\chi)(Y + \chi)^2} \left(\partial_\mu \text{Im} \hat{S} - 4 \text{Re} \Phi \partial_\mu \text{Im} \Phi \right)^2,\end{aligned}\quad (3.46)$$

where

$$\hat{S} = S + \Phi^2, \quad (3.47)$$

for which $Y = \hat{S} + \bar{\hat{S}} - (\Phi + \bar{\Phi})^2$ and $\text{Im}(dS - 2\bar{\Phi} d\Phi) = d \text{Im} \hat{S} - 4 \text{Re} \Phi d \text{Im} \Phi$. From the existence of solutions (3.44) and positivity of the Lagrangian, the range of Y is $Y + \chi > 0$ if $\chi > 0$ and $Y + 3\chi > 0$ if $\chi < 0$. Writing as usual

$$\mathcal{L} = \frac{1}{\kappa^2} g_{ab} (\partial_\mu q^a) (\partial^\mu q^b) = G_{ab} (\partial_\mu q^a) (\partial^\mu q^b), \quad (3.48)$$

$q^a = (Y, \text{Re} \Phi, \text{Im} \Phi, \text{Im} \hat{S})$, and comparing $ds^2 = g_{ab} dq^a dq^b$ with expression (3.4), we see that the hypermultiplet kinetic metric g_{ab} is the CPH metric with

$$Y = V - 2\chi = \rho^2 - 2\chi, \quad (3.49)$$

and with¹⁴

$$\Phi = \frac{1}{\sqrt{2}} (\eta + i\varphi), \quad \text{Im} \hat{S} = -2\tau. \quad (3.50)$$

Positivity of kinetic terms is obtained if $V = \rho^2 > |\chi|$ which is, as explained at the end of Subsection 3.1, the natural domain of V .

¹³ All fields and parameter χ are dimensionless.

¹⁴ This choice is not unique. We may for instance rotate Φ using isometry M .

As already observed, the case $\chi = 0$ corresponds to the $SU(2,1)/SU(2) \times U(1)$ metric

$$ds^2 = \frac{1}{Y^2} \left[\frac{1}{4} dY^2 + \left(d \operatorname{Im} \hat{S} - 4 \operatorname{Re} \Phi d \operatorname{Im} \Phi \right)^2 \right] + \frac{2}{Y} d\Phi d\bar{\Phi}. \quad (3.51)$$

With Kähler coordinates \hat{S} and Φ , the Kähler potential is $K = -\ln Y$, with $Y = V = \hat{S} + \bar{\hat{S}} - (\Phi + \bar{\Phi})^2$.

This relatively simple construction of the one-loop-corrected dilaton hypermultiplet metric allows easily to derive the full $N = 2$ supergravity Lagrangian, using $N = 2$ superconformal tensor calculus [13, 8, 9].

4 Zero-curvature hyper-Kähler limit

All quaternion-Kähler metrics are Einstein spaces with nonzero curvature. With one hypermultiplet, the scalar kinetic Lagrangian (3.48) verifies [1]

$$R_{ab} = -6 g_{ab} = -6\kappa^2 G_{ab}. \quad (4.1)$$

The link with global $N = 2$ supersymmetry is realized by defining a $\kappa \rightarrow 0$ hyper-Kähler limit of the CPH metric (3.4) or (3.46) in which, if feasible, the Heisenberg algebra does not contract to an abelian symmetry. As observed in Subsection 3.1, the magnitude of χ can be eliminated by rescaling of the coordinates (in the absence of D-branes). We then have three $|\chi|$ -independent cases to examine: firstly, positive χ , with $V > 0$; secondly, $\chi = 0$ ($V > 0$) which is $SU(1,2)/SU(2) \times U(1)$; thirdly, a negative χ , with $V > |\chi|$. In each case, we should seek to find a parameter-free zero-curvature limit. The most interesting case turns out to be χ negative, which we first study.

With χ negative, we are interested in the CPH metric in the region $V + \chi \sim 0$. We then apply to metric (3.4) the following change of variables:

$$\begin{aligned} V &= 2|\chi| \kappa^{2/3} \mu^{-1/3} C - \chi, & \varphi &= \sqrt{|\chi|} \kappa^{2/3} \mu^{-1/3} \hat{\varphi}, \\ \eta &= \sqrt{|\chi|} \kappa^{2/3} \mu^{-1/3} \hat{\eta}, & \tau &= |\chi| \kappa^{4/3} \mu^{1/3} \hat{\tau}, \end{aligned} \quad (4.2)$$

where μ is an arbitrary mass scale. Positivity of the metric, $V + \chi > 0$ implies $C > 0$. While the original fields are dimensionless, the new, hatted, fields $(C, \hat{\phi}, \hat{\eta}, \hat{\tau})$ have canonical dimension. With this choice of dependence in κ , the resulting metric is

$$\begin{aligned} ds^2 = g_{ab} dq^a dq^b &= \frac{\kappa^2}{2} \frac{\mu C}{[(\kappa\mu)^{2/3} C + \mu]^2} \left[\frac{dC^2}{2\kappa^{2/3} \mu^{-1/3} C + 1} + d\hat{\eta}^2 + d\hat{\varphi}^2 \right] \\ &+ \frac{\kappa^2 \mu^2}{2C} \frac{2(\kappa\mu)^{2/3} C + \mu}{[(\kappa\mu)^{2/3} C + \mu]^2} \left[d\hat{\tau} + \frac{1}{\mu} \hat{\eta} d\hat{\varphi} \right]^2, \end{aligned} \quad (4.3)$$

since $\chi = -|\chi|$. Using this metric in Lagrangian (3.48), the overall factor κ^2 cancels and we can take the limit $\kappa \rightarrow 0$, with result

$$\mathcal{L}_{\kappa \rightarrow 0} = \frac{C}{2\mu} [(\partial_\mu C)^2 + (\partial_\mu \hat{\eta})^2 + (\partial_\mu \hat{\varphi})^2] + \frac{\mu}{2C} \left[\partial_\mu \hat{\tau} + \frac{1}{\mu} \hat{\eta} \partial_\mu \hat{\varphi} \right]^2. \quad (4.4)$$

This scalar Lagrangian has the hyper-Kähler metric with Heisenberg symmetry (2.20) with $A = 1/\mu$ and $B = 0$ and with relations $\Phi = \frac{1}{\sqrt{2}}(\hat{\eta} + i\hat{\varphi})$, $\hat{\tau} = 2\tau$. As noticed earlier, parameter B can always be absorbed in a shift of C , as long as $A \neq 0$.

Notice that to obtain limit (4.4), we only need the change of variables (4.2) up to higher orders in κ . In particular, according to Eq. (3.9), we may write the four-dimensional string dilaton as

$$\begin{aligned} e^{-2\phi_4} &= 2|\chi|\kappa^{2/3}\mu^{-1/3}C - 2\chi, \\ \phi_4 &= \langle \phi_4 \rangle - \kappa^{2/3}\mu^{-1/3}\hat{\phi}_4, \\ e^{-2\langle \phi_4 \rangle} &= -2\chi = 2|\chi|, & C &= 2\hat{\phi}_4, \end{aligned} \quad (4.5)$$

in terms of the fluctuation $\hat{\phi}_4$ and of the background value $\langle \phi_4 \rangle$. Since $|\chi| = \chi_1 = \chi_E/(12\pi)$, we are considering the case of a positive Euler number $\chi_E = 2(h_{11} - h_{21})$, with h_{11}, h_{12} the corresponding Betti numbers of the CY_3 manifold. A typical example with a single hypermultiplet would be IIA strings on a CY_3 manifold with $h_{21} = 0$. Positivity-related questions with several hypermultiplets, as is in particular the case with a negative Euler number, should be reanalyzed.

Comparing the scalings (4.2) and the identification of the string coupling in the last Eq. (4.5), we see that the R-R fields η and φ carry as expected a supplementary factor g_{string} .

We could also consider the single-tensor version of the theory. Dualizing $\hat{\tau}$ into $H_{\mu\nu\rho}$, we find

$$\begin{aligned} \mathcal{L}_{\kappa \rightarrow 0, ST} &= \frac{C}{\mu} \left[\frac{1}{2}(\partial_\mu C)^2 + \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + (\partial_\mu \bar{\Phi})(\partial^\mu \Phi) \right] \\ &\quad - \frac{i}{12\mu} \epsilon^{\mu\nu\rho\sigma} (\bar{\Phi} \partial_\mu \Phi - \Phi \partial_\mu \bar{\Phi}) H_{\nu\rho\sigma}. \end{aligned} \quad (4.6)$$

This is the bosonic sector (2.8) of the single-tensor theory (2.7) with again $A = 1/\mu$ and $B = 0$. Then, for negative χ , the $N = 2$ supergravity hypermultiplet with Heisenberg symmetry is described in the global supersymmetry limit by the unique nontrivial theory with the same symmetry.

For completeness, we may also consider the case of the CPH metric with positive

χ . The interesting limiting regions are $V \sim 0$ and $V - \chi \sim 0$. If $V = \rho^2 \ll \chi$,

$$ds_{CPH}^2 = \frac{1}{\chi}(d\rho^2 + d\eta^2 + d\varphi^2) + \frac{4\rho^2}{\chi^3}(d\tau + \eta d\varphi)^2. \quad (4.7)$$

The appropriate rescalings are $(\rho, \eta, \varphi, \tau) = (\sqrt{\chi}\kappa\hat{\rho}, \sqrt{\chi}\kappa\hat{\eta}, \sqrt{\chi}\kappa\hat{\varphi}, \chi\hat{\tau})$ to obtain

$$ds_{CPH}^2 = \kappa^2 [d\hat{\rho}^2 + d\hat{\eta}^2 + d\hat{\varphi}^2 + 4\hat{\rho}^2(d\hat{\tau} + \kappa^2\hat{\eta}d\hat{\varphi})^2]. \quad (4.8)$$

The Heisenberg symmetry acting on the rescaled fields has algebra $[X, Y] = 2\kappa^2 Z$. In the limit $\kappa \rightarrow 0$, it contracts to $[X, Y] = 0$ and we find

$$\lim_{\kappa \rightarrow 0} \frac{1}{\kappa^2} ds_{CPH}^2 = d\hat{\rho}^2 + 4\hat{\rho}^2 d\hat{\tau}^2 + d\hat{\eta}^2 + d\hat{\varphi}^2, \quad (4.9)$$

which is the trivial four-dimensional euclidean space. The second region of interest if $\chi > 0$ is $V - \chi \sim 0$. First, we change coordinates to

$$V = 2\lambda C + \chi, \quad \eta = \lambda\hat{\eta}/\sqrt{\chi}, \quad \varphi = \lambda\hat{\varphi}/\sqrt{\chi}, \quad \tau = \lambda\hat{\tau} \quad (4.10)$$

and the metric for $\lambda \rightarrow 0$ and χ finite reads

$$ds_{CPH}^2 = \frac{1}{2C^2} [dC^2 + d\hat{\eta}^2 + d\hat{\varphi}^2 + d\hat{\tau}^2]. \quad (4.11)$$

This limiting metric is $SO(1,4)/SO(4)$, again with $R_{ij} = -6g_{ij}$ and with radius $\sim \langle C \rangle$. In the large radius, zero-curvature limit, the metric is trivial. Finally, in the $SU(1,2)/SU(2) \times U(1)$ case $\chi = 0$, the zero-curvature limit is again trivial.

The conclusion is that in the zero-curvature limit, the CPH one-loop Lagrangian for the dilaton hypermultiplet is the hyper-Kähler $N = 2$ sigma-model with Heisenberg symmetry (2.7). If the one-loop parameter χ is negative, then $A \neq 0$ and the Heisenberg algebra has a non-trivial realization in this limit. If $\chi \geq 0$ however, $A = 0$ and the limit of $N = 2$ global supersymmetry is the free hypermultiplet. In the string context, the above non-trivial limit can be taken if the string coupling is tuned at a fixed value, according to the third line of Eq. (4.5), which applies with positive Euler number.

In a recent paper [5], we have constructed the interaction of a hypermultiplet with the Dirac-Born-Infeld Maxwell Lagrangian. The hypermultiplet sector has a full linear $N = 2$ supersymmetry while the second supersymmetry is nonlinearly realized on the Maxwell superfield W_α . As an application of our results, we can easily use our identification of the string universal hypermultiplet. The bosonic DBI action, after

elimination of the Maxwell auxiliary field and using the single-tensor formulation, is¹⁵

$$\begin{aligned}\mathcal{L}_{DBI} = & \frac{1}{8f}(2g\text{Re}\Phi - \frac{1}{f}) \left[1 - \sqrt{1 + \frac{2g^2 C^2}{(2g\text{Re}\Phi - \frac{1}{f})^2}} \sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}f F_{\mu\nu})} \right] \\ & + g\epsilon^{\mu\nu\rho\sigma} \left(\frac{f}{4}\text{Im}\Phi F_{\mu\nu}F_{\rho\sigma} - \frac{1}{4}B_{\mu\nu}F_{\rho\sigma} + \frac{1}{24f}C_{\mu\nu\rho\sigma} \right).\end{aligned}\tag{4.12}$$

In this expression, f is the breaking scale of the second, nonlinearly realized supersymmetry (with dimension (energy)⁻²) and g is the Chern-Simons coupling¹⁶ (equal to the string coupling for a D3-brane). The four-form field $C_{\mu\nu\rho\sigma}$ is a component of the single-tensor multiplet required by supersymmetry of the nonlinear theory [5].

Since we have control of the kinetic Lagrangian of the universal string hypermultiplet in the global supersymmetry limit, we can then identify the single-tensor fields in terms of string fields. First, C is the global dilaton and $B_{\mu\nu}$ is the NS-NS tensor. Then, the complex scalar Φ includes the R-R fields. The supersymmetric minimum of the scalar potential included in theory (4.12) implies $\langle C \rangle = 0$ and Φ corresponds to flat directions of this vacuum.

5 Conclusions

In this work, we analyzed the effective field theory of the universal dilaton hypermultiplet of type II string compactifications on a CY₃ manifold with a special emphasis on the global supersymmetry limit. The perturbative isometries form the two-dimensional Euclidean algebra E_2 with a central extension, which contains a Heisenberg subalgebra. Using this isometry as a guiding principle and the method of quaternionic quotient in conformal supergravity, we rederived the two-derivative $N = 2$ supergravity action, depending on a deformation parameter that corresponds to the one-loop correction proportional to the Euler number of the CY₃ manifold. We then established the precise connection with a one-parameter family of hyper-Kähler spaces in $N = 2$ global supersymmetry, possessing the same isometry, by defining a non-trivial gravity decoupled limit characterized by a new mass scale. This requires the string coupling to be tuned at a fixed value which only occurs for a positive Euler number. As the latter becomes large, the theory becomes weakly coupled, justifying the perturbative approximation

¹⁵In Ref. [5], this is the electric version of the theory, induced by a $N = 2$ Chern-Simons coupling $gB \wedge F$.

¹⁶In contrast to Ref. [5], we have defined single-tensor fields with canonical dimension so that g has dimension (energy). We also chose the Fayet-Iliopoulos term to be $1/\mathcal{F}$ so that gauge kinetic terms are canonically normalized at $\text{Re}\Phi = 0$.

in taking the global limit. Notice that in the absence of moduli stabilization effects, this positivity requirement is compatible with the possibility of choosing $h_{21} = 0$ in type IIA, that guarantees the absence of other hypermultiplets in the spectrum, which could modify the constraints we derived from the positivity of the metric. It would be interesting to understand this requirement in the general case.

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References

- [1] J. Bagger and E. Witten, Nucl. Phys. B **222** (1983) 1.
- [2] S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A **4** (1989) 2475;
S. Cecotti, Commun. Math. Phys. **124** (1989) 23;
S. Ferrara and S. Sabharwal, Class. Quant. Grav. **6** (1989) L77.
- [3] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, Class. Quant. Grav. **20**, 5079 (2003) [arXiv:hep-th/0307268].
- [4] I. Antoniadis, S. Ferrara, R. Minasian and K. S. Narain, Nucl. Phys. B **507** (1997) 571 [arXiv:hep-th/9707013].
- [5] N. Ambrosetti, I. Antoniadis, J.-P. Derendinger and P. Tziveloglou, Nucl. Phys. B **835** (2010) 75 [arXiv:0911.5212].
- [6] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. **80** (1981) 443.
- [7] D. M. J. Calderbank and H. Pedersen, J. Diff. Geom. **60** 485 [arXiv:math/0105263].
- [8] K. Galicki, Nucl. Phys. B **271** (1986) 402.
- [9] K. Galicki, Nucl. Phys. B **289** (1987) 573.

- [10] U. Lindström and M. Roček, Nucl. Phys. B **222** (1983) 285.
- [11] W. Siegel, Phys. Lett. B **85** (1979) 333.
- [12] S. W. Hawking, Phys. Lett. A **60** (1977) 81;
G. W. Gibbons and S. W. Hawking, Phys. Lett. B **78** (1978) 430.
- [13] B. de Wit, J. W. van Holten and A. Van Proeyen, Nucl. Phys. B **167** (1980) 186;
B. de Wit, P. G. Lauwers and A. Van Proeyen, Nucl. Phys. B **255** (1985) 569.
- [14] L. Anguelova, M. Rocek and S. Vandoren, Phys. Rev. D **70**, 066001 (2004)
[arXiv:hep-th/0402132].
- [15] P. Y. S. Casteill, E. Ivanov and G. Valent, Nucl. Phys. B **627** (2002) 403
[arXiv:hep-th/0110280].